Automatic Control (1)

By

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Lecture (4)

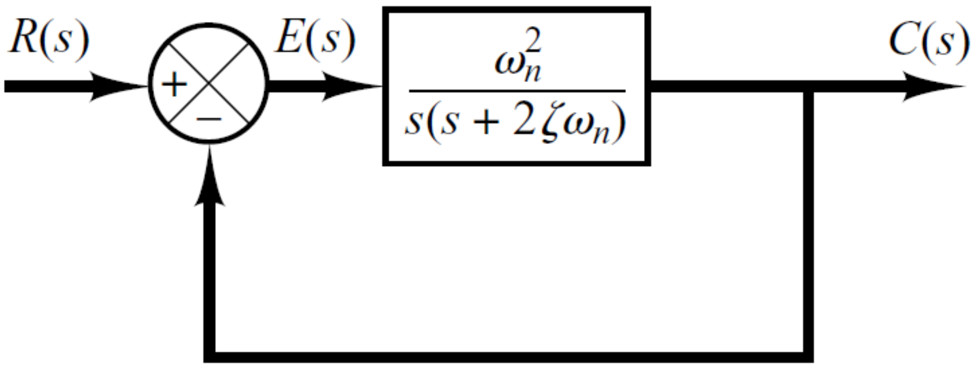
Second Order System

- We have already discussed the affect of location of poles and zeros on the transient response of 1^{st} order systems.
- Compared to the simplicity of a first-order system, a second-order system exhibits a wide range of responses that must be analyzed and described.
- Varying a first-order system's parameter (T, K) simply changes the speed and offset of the response
- Whereas changes in the parameters of a second-order system can change the *form of* the response.
- A second-order system can display characteristics much like a first-order system or, depending on component values, display damped or pure oscillations for its *transient response*.

Introduction

- A general second-order system is characterized by the following transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



ω_n ⟶ un-damped natural frequency of the second order system, which is the frequency of oscillation of the system without damping.

ζ ⟶ damping ratio of the second order system, which is a measure of the degree of resistance to change in the system output.

Example 2

- Determine the un-damped natural frequency and damping ratio of the following second order system.

$$\frac{C(s)}{R(s)} = \frac{4}{s^2 + 2s + 4}$$

- Compare the numerator and denominator of the given transfer function with the general 2nd order transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega_n^2 = 4 \qquad \Rightarrow \omega_n = 2$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 2s + 4$$

$$\Rightarrow 2\zeta\omega_n s = 2s$$

$$\Rightarrow \zeta\omega_n = 1$$

$$\Rightarrow \zeta = 0.5$$

Introduction

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- Two poles of the system are

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$
$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

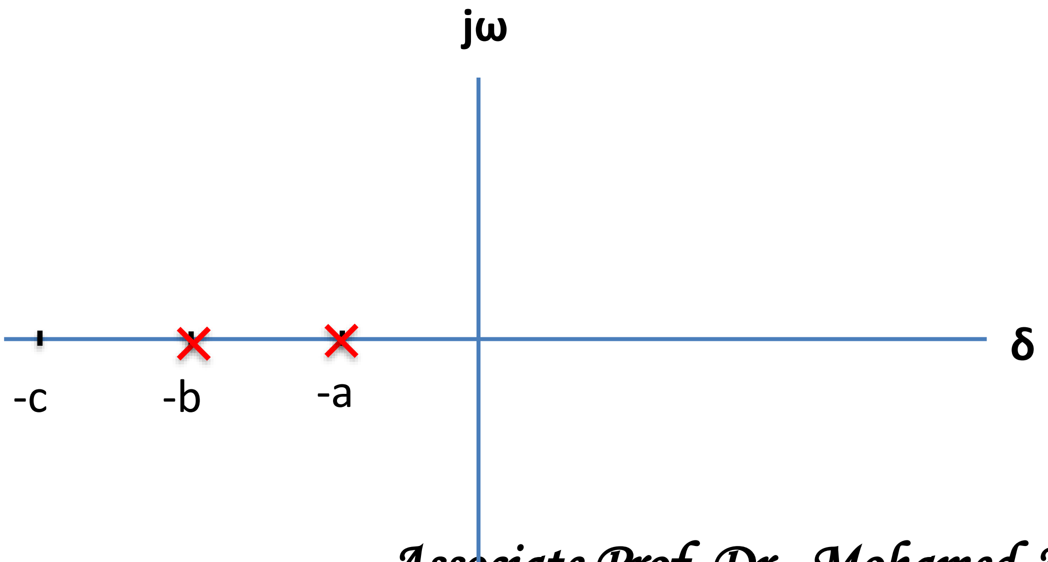
Introduction

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

- According the value of ζ , a second-order system can be set into one of the four categories:

  1. Overdamped - when the system has two real distinct poles (ζ >1).



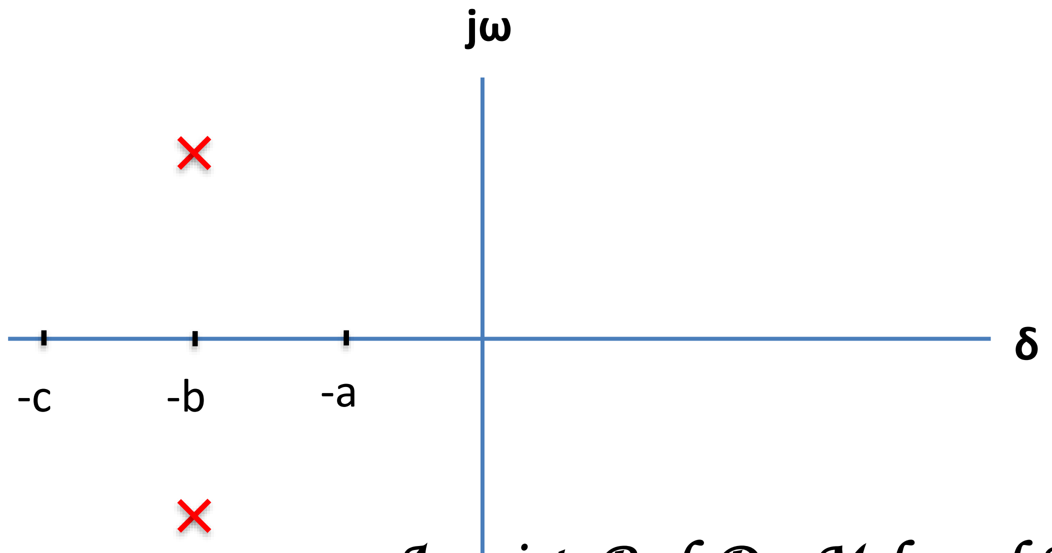
Introduction

$$-\omega_n\zeta+\omega_n\sqrt{\zeta^2-1}$$

$$-\omega_n\zeta-\omega_n\sqrt{\zeta^2-1}$$

- According the value of ζ , a second-order system can be set into one of the four categories:

2. *Underdamped* - when the system has two complex conjugate poles ($0<\zeta<1$)



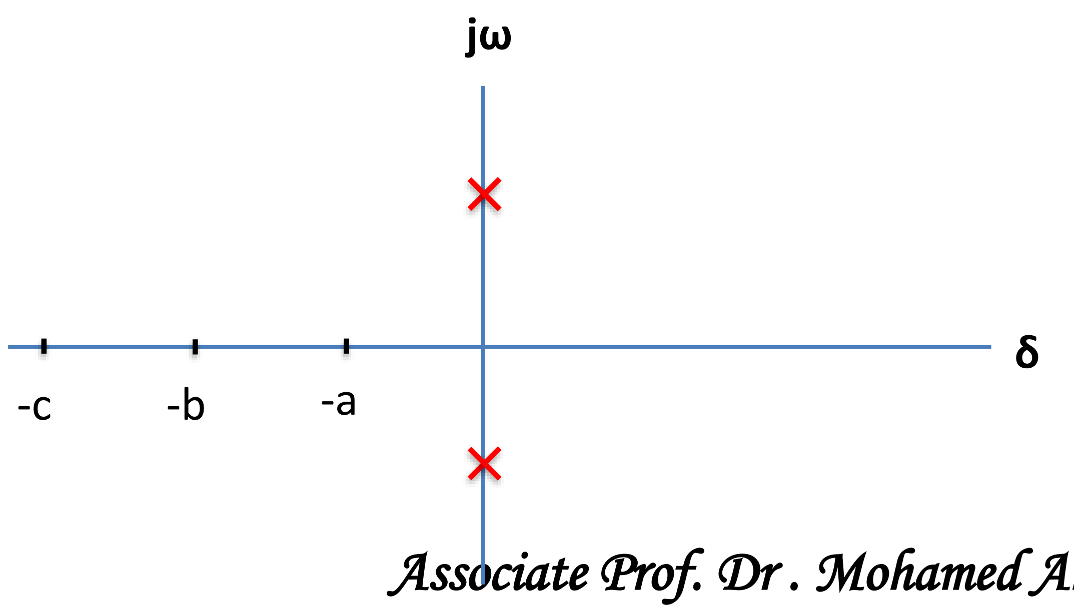
Introduction

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

- According the value of ζ , a second-order system can be set into one of the four categories:

3. *Undamped* - when the system has two imaginary poles (ζ = 0).



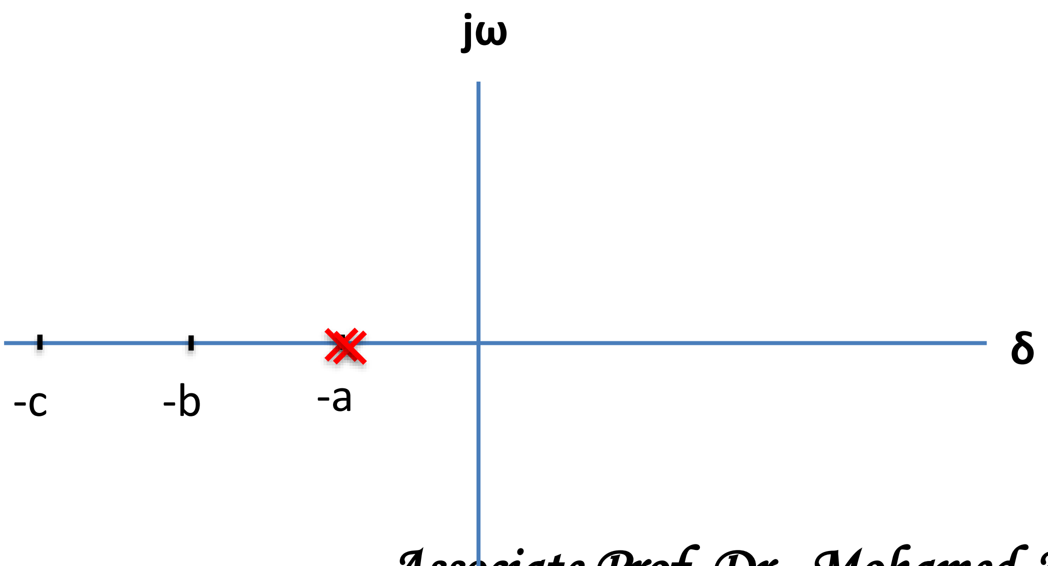
Introduction

$$-\omega_n\zeta+\omega_n\sqrt{\zeta^2-1}$$

$$-\omega_n\zeta-\omega_n\sqrt{\zeta^2-1}$$

- According the value of ζ , a second-order system can be set into one of the four categories:

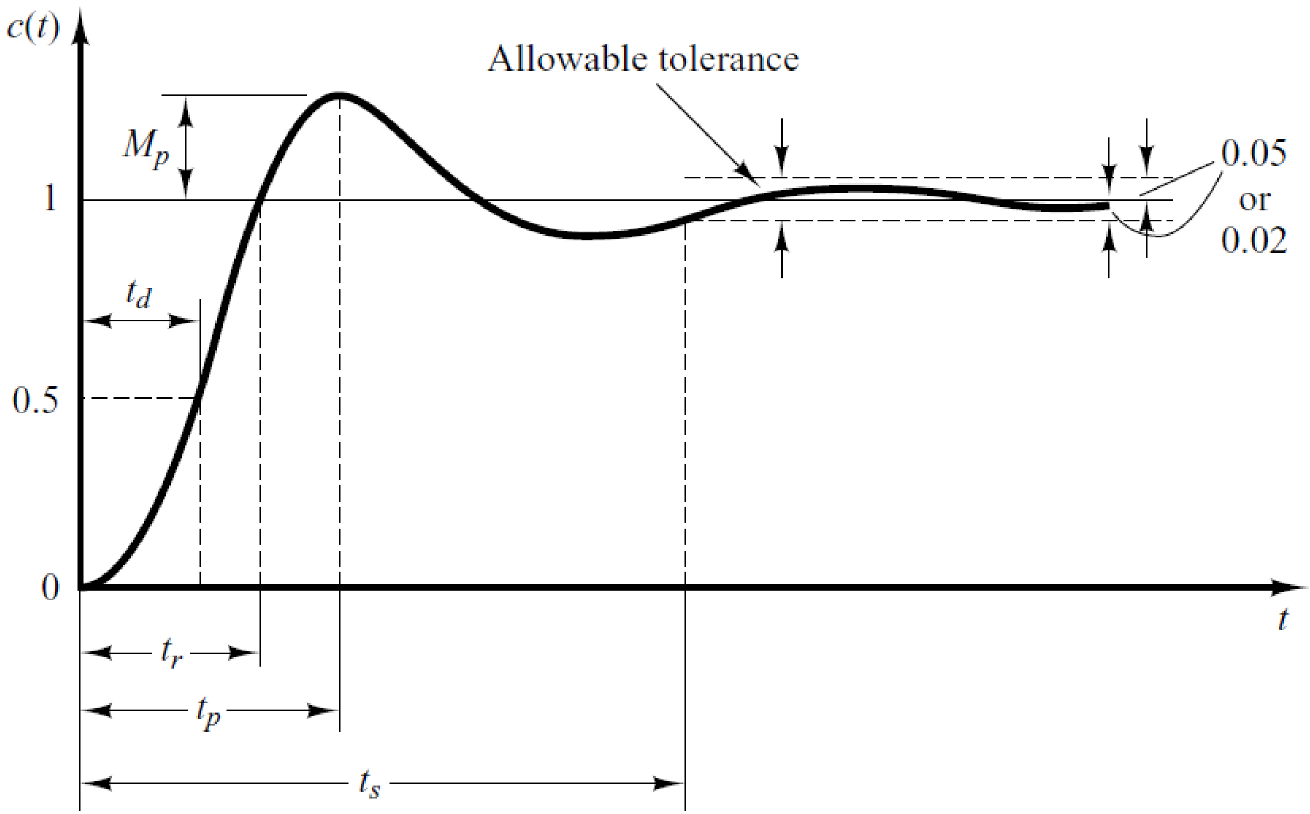
4. *Critically damped* - when the system has two real but equal poles (ζ = 1).



Underdamped System

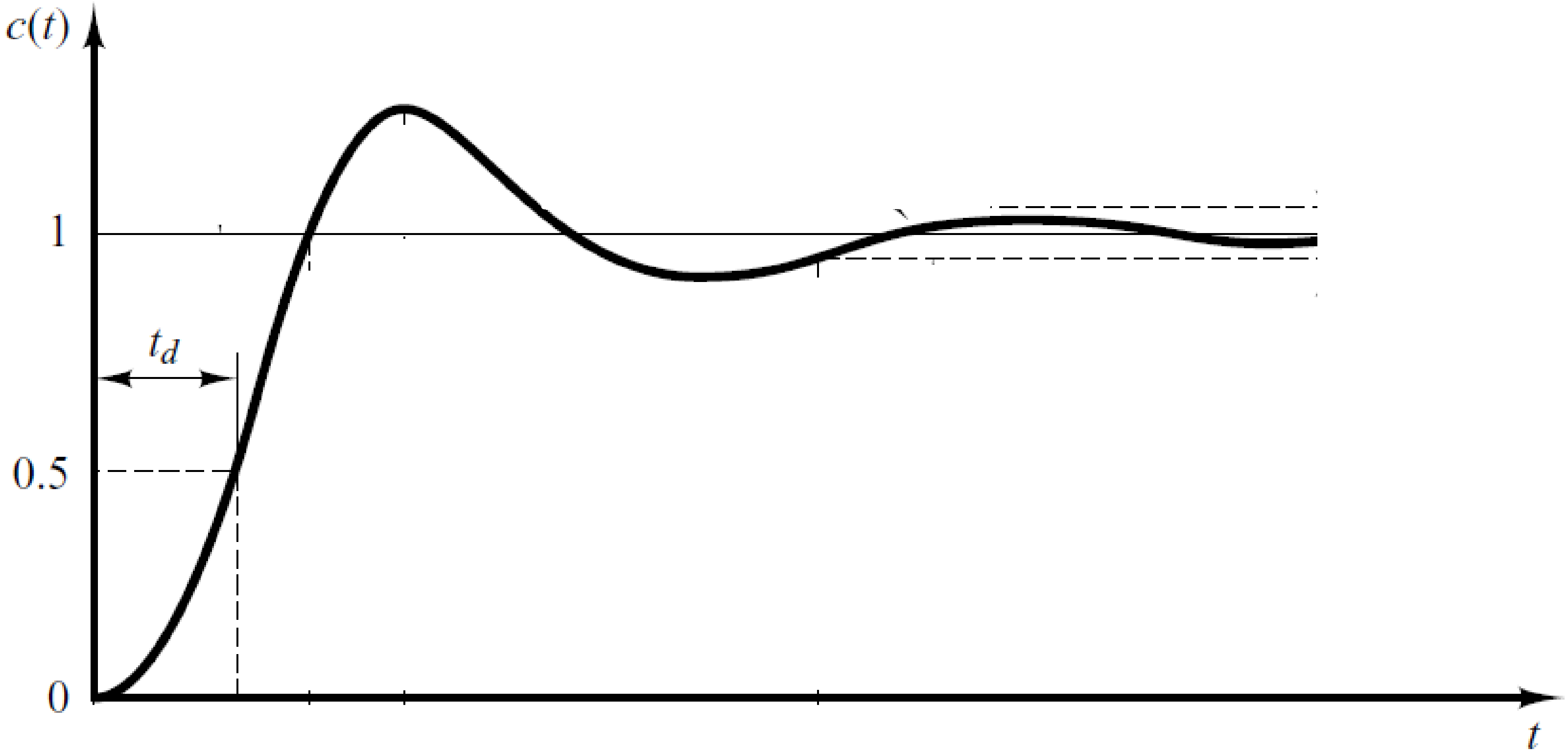
For $0 < \zeta < 1$ and $\omega_n > 0$, the 2^{nd} order system's response due to a unit step input is as follows.

Important timing characteristics: delay time, rise time, peak time, maximum overshoot, and settling time.



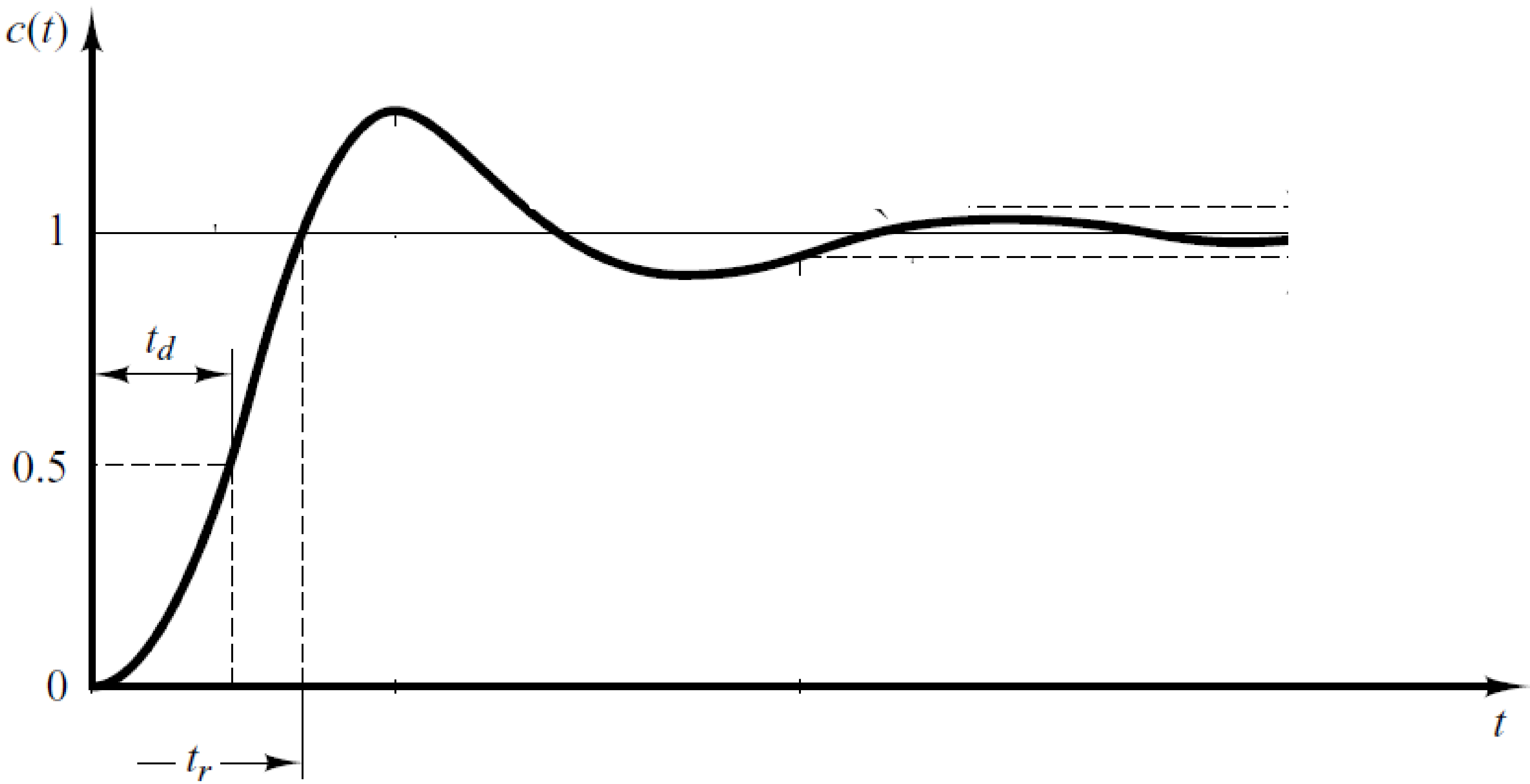
Delay Time

- The delay (t_d) time is the time required for the response to reach half the final value the very first time.



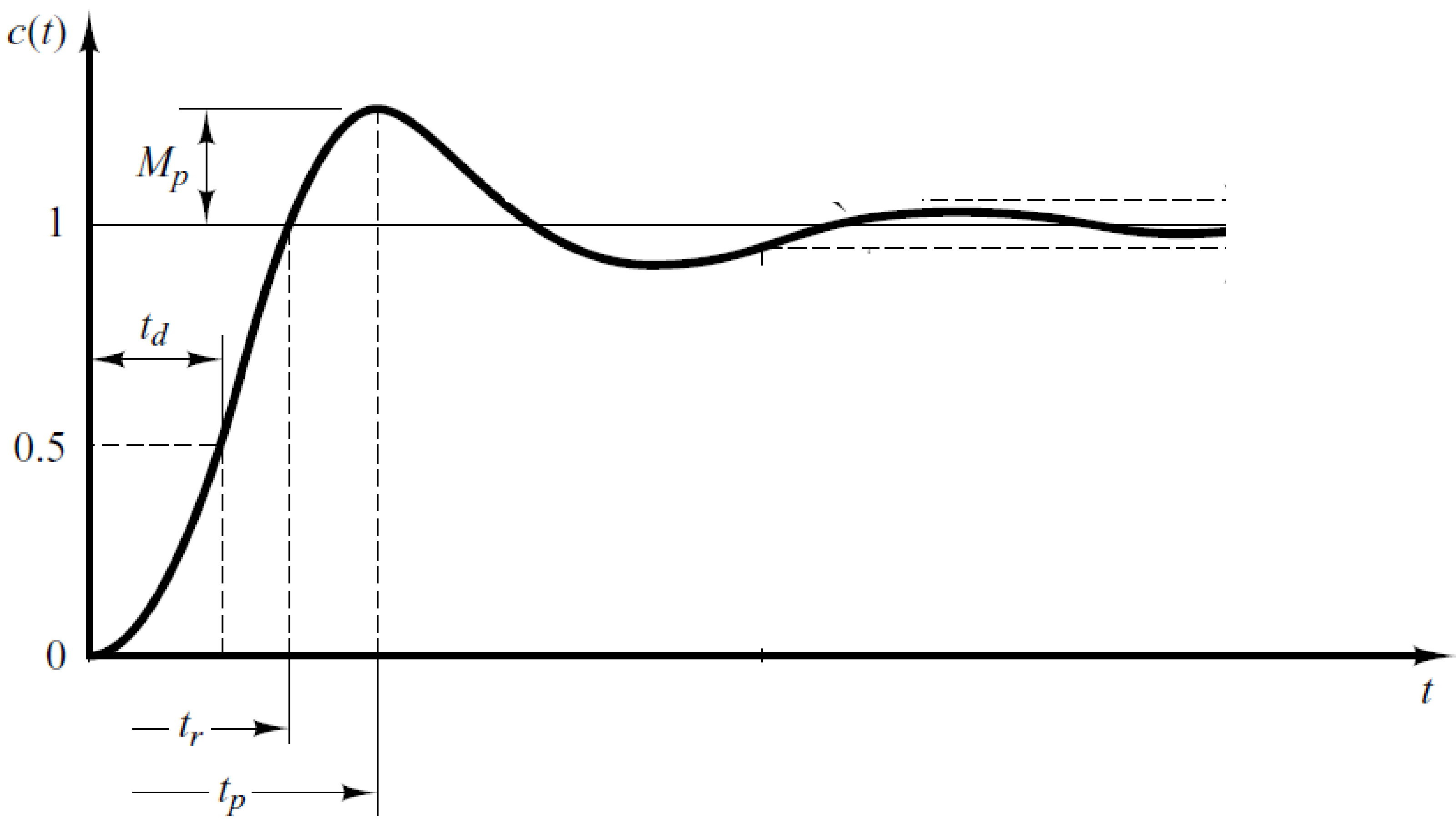
Rise Time

- The rise time is the time required for the response to rise from 10% to 90%, 5% to 95%, or 0% to 100% of its final value.
- For underdamped second order systems, the 0% to 100% rise time is normally used. For overdamped systems, the 10% to 90% rise time is commonly used.



Peak Time

- The peak time is the time required for the response to reach the first peak of the overshoot.



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Maximum Overshoot

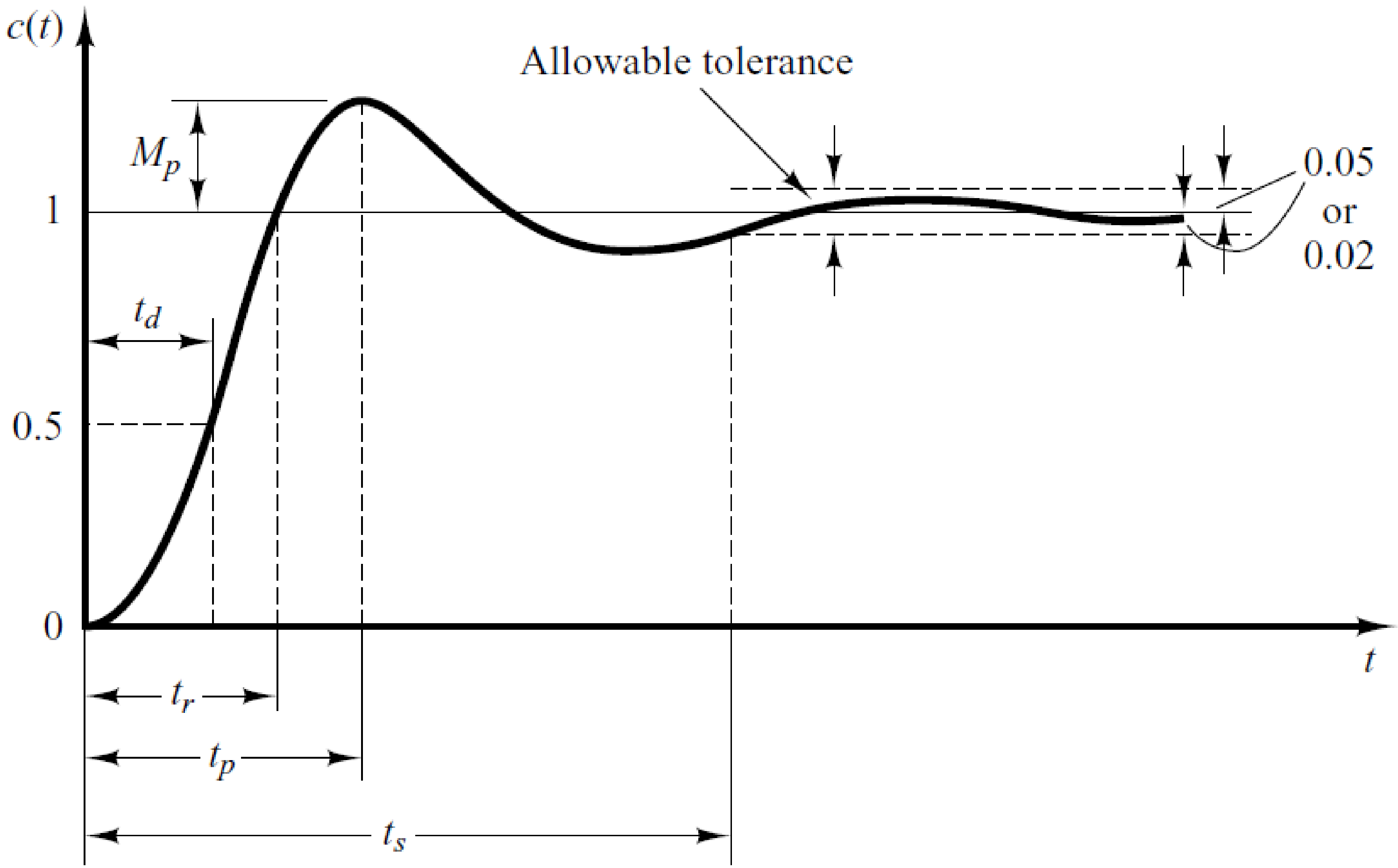
The maximum overshoot is the maximum peak value of the response curve measured from unity. If the final steady-state value of the response differs from unity, then it is common to use the maximum percent overshoot. It is defined by

$$\text{Maximum percent overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

The amount of the maximum (percent) overshoot directly indicates the relative stability of the system.

Settling Time

- The settling time is the time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value (usually 2% or 5%).



Step Response of underdamped System

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \xrightarrow{\text{Step Response}} C(s) = \frac{\omega_n^2}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)}$$

- The partial fraction expansion of above equation is given as

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega_n^2\left(1 - \zeta^2\right)$$

$$\left(s + 2\zeta\omega_n\right)^2 \qquad C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \zeta^2\omega_n^2 + \omega_n^2 - \zeta^2\omega_n^2}$$

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{\left(s + \zeta\omega_n\right)^2 + \omega_n^2\left(1 - \zeta^2\right)}$$

Step Response of underdamped System

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)}$$

- Above equation can be written as

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

- Where $\omega_d = \omega_n\sqrt{1 - \zeta^2}$, is the frequency of transient oscillations and is called damped natural frequency.

- The inverse Laplace transform of above equation can be obtained easily if C(s) is written in the following form:

$$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

Step Response of underdamped System

$$C(s)=\frac{1}{s}-\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}-\frac{\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}$$

$$C(s)=\frac{1}{s}-\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}-\frac{\frac{\zeta}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2+\omega_d^2}$$

$$C(s)=\frac{1}{s}-\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}-\frac{\zeta}{\sqrt{1-\zeta^2}}\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}$$

$$c(t)=1-e^{-\zeta\omega_n t}\cos\omega_d t-\frac{\zeta}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\omega_d t$$

Step Response of underdamped System

$$c(t) = 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin\omega_d t$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

- When $\zeta = 0$

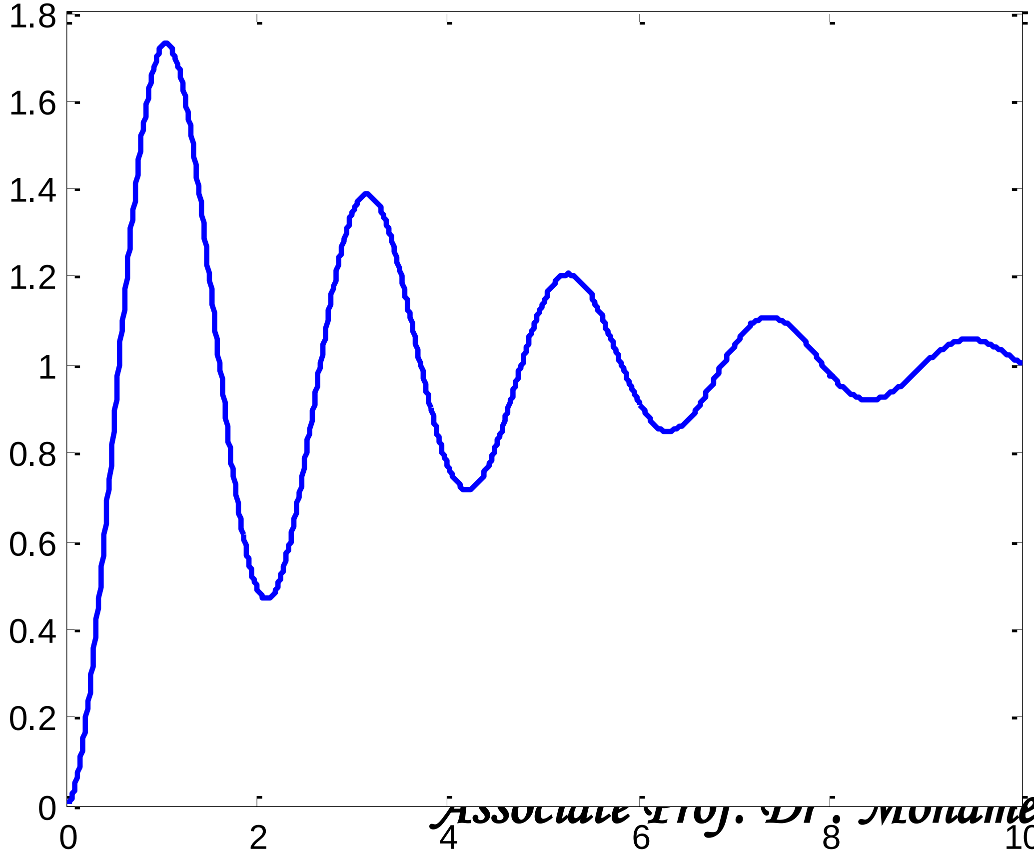
$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$
$$= \omega_n$$

$$c(t) = 1 - \cos\omega_n t$$

Step Response of underdamped System

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

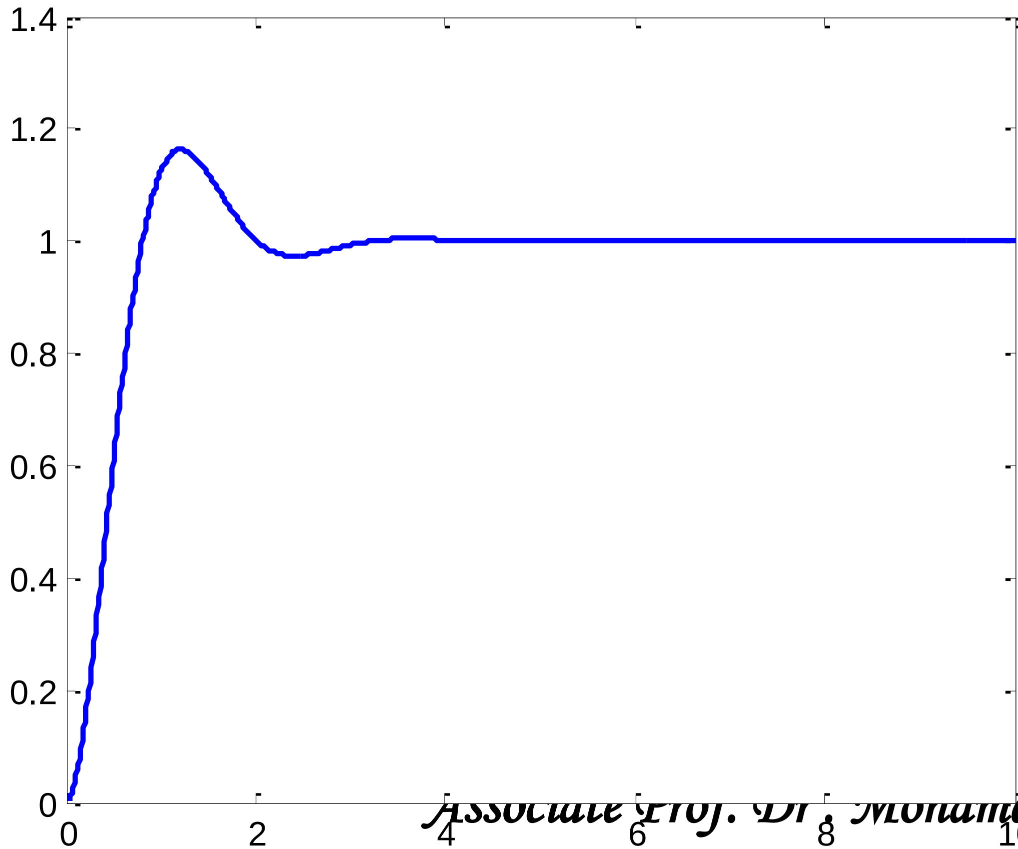
if $\zeta = 0.1$ and $\omega_n = 3$



Step Response of underdamped System

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

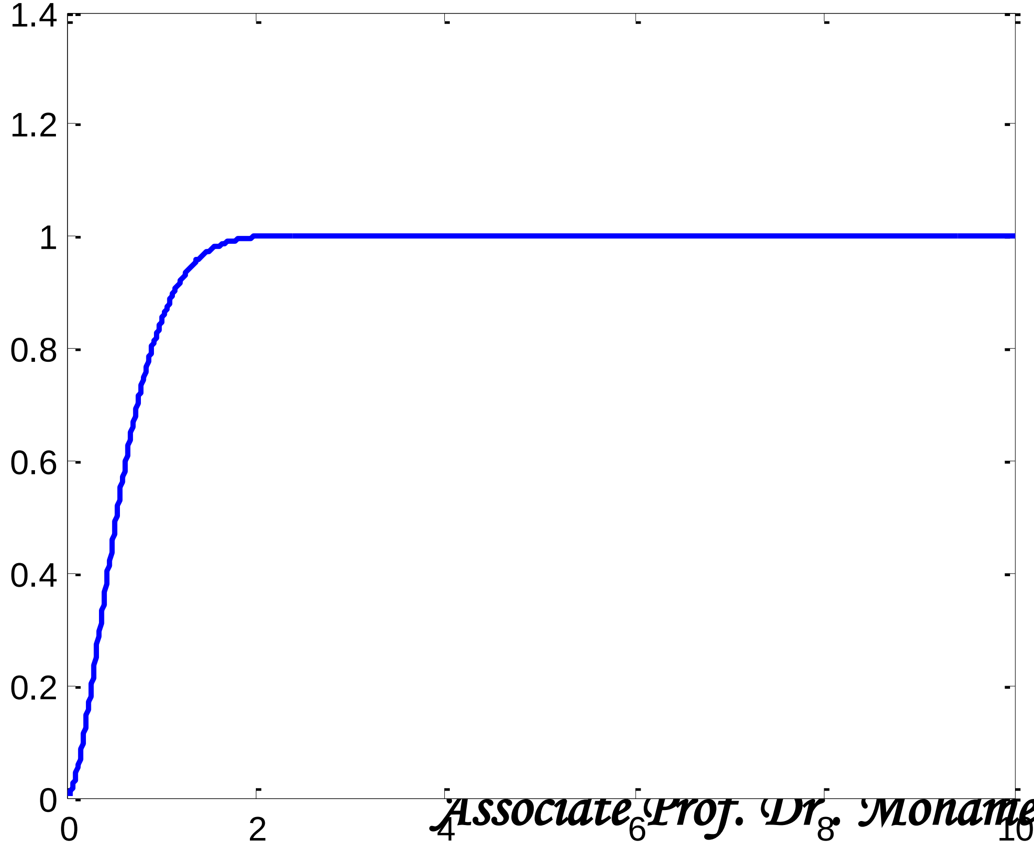
if $\zeta = 0.5$ and $\omega_n = 3$



Step Response of underdamped System

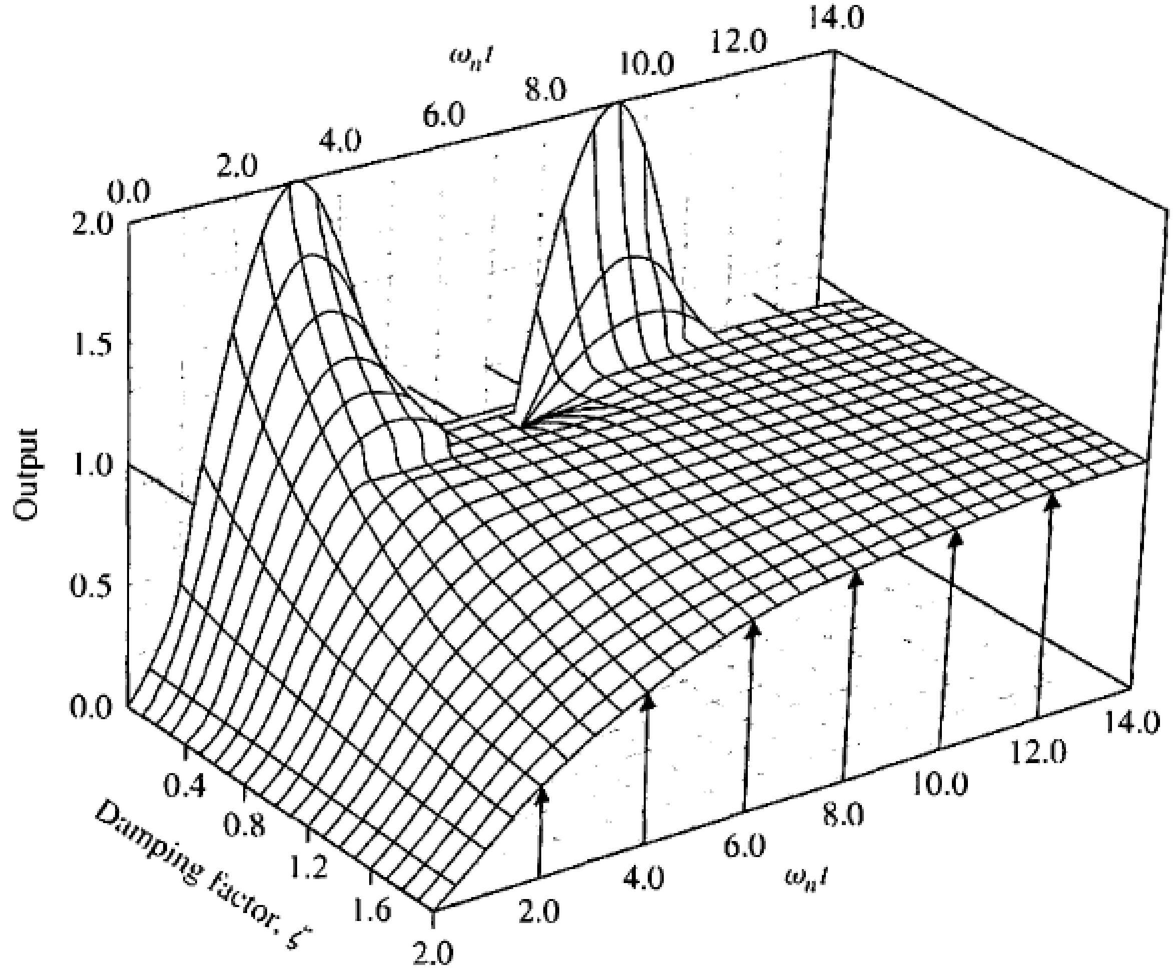
$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

if $\zeta = 0.9$ and $\omega_n = 3$



Step Response of underdamped System

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

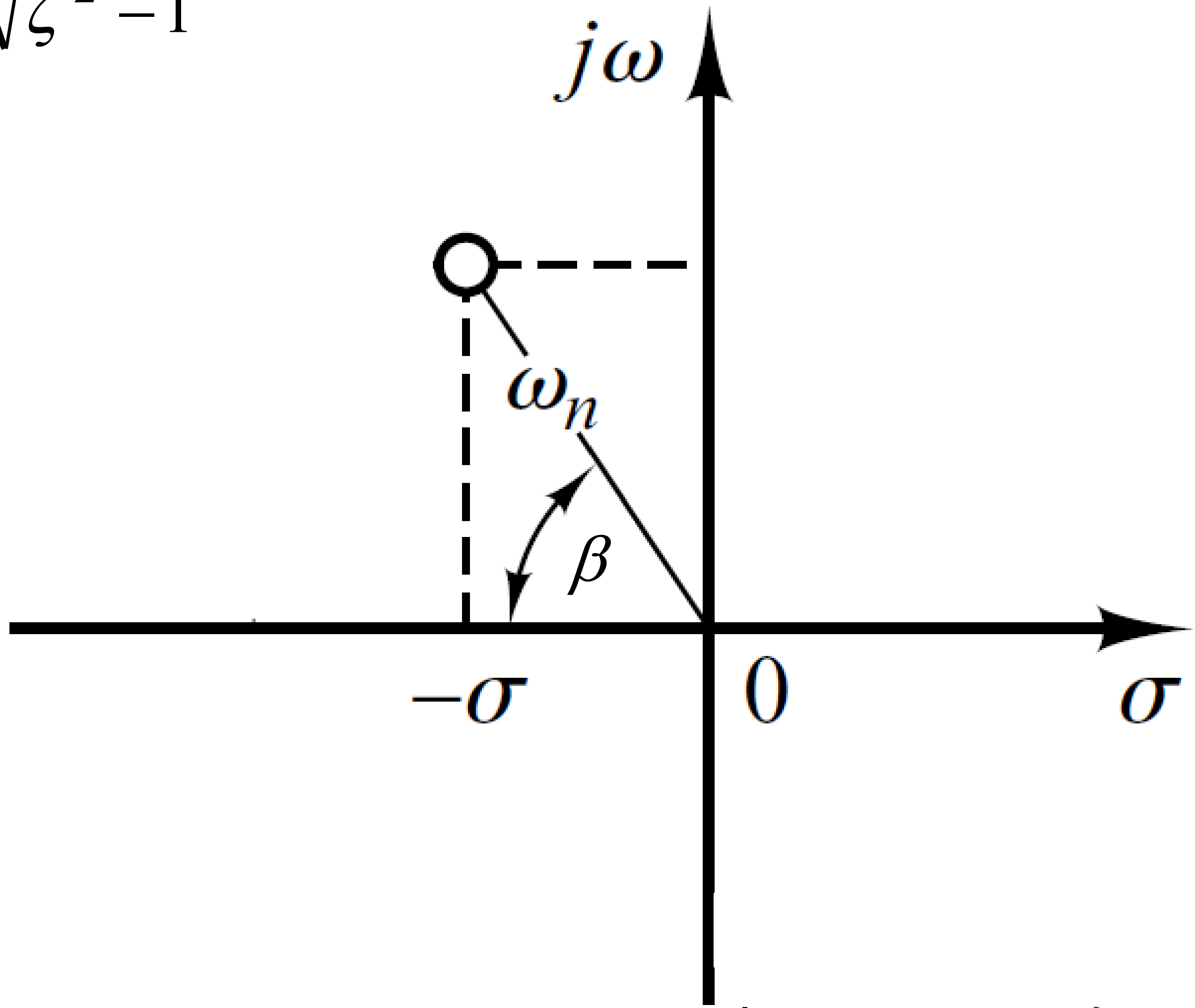


S-Plane (Underdamped System)

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

Since $\omega^2\zeta^2 - \omega^2(\zeta^2 - 1) = \omega^2$, the distance from the pole to the origin is ω and $\zeta = cos\beta$



Analytical Solution

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

- Rise time: set c(t)=1, we have $t_r = \frac{\pi-\beta}{\omega_d}$ $\quad \omega_d = \omega_n\sqrt{1-\zeta^2}$
- Peak time: set $\frac{dc(t)}{dt} = 0$, we have $t_p = \frac{\pi}{\omega_d}$
- Maximum overshoot: $\mathrm{M}_p = c(t_p) - 1$ $= e^{-(\zeta\omega/\omega_d)\pi}$ (for unity output)
- Settling time: the time for the outputs always within 2% of the final value is approximately $\frac{4}{\zeta\omega}$

Steady State Error

- If the output of a control system at steady state does not exactly match with the input, the system is said to have steady state error.
- Any physical control system inherently suffers steady-state error in response to certain types of inputs.
- A system may have no steady-state error to a step input, but the same system may exhibit nonzero steady-state error to a ramp input.

Classification of Control Systems

- Control systems may be classified according to their ability to follow step inputs, ramp inputs, parabolic inputs, and so on.
- The magnitudes of the steady-state errors due to these individual inputs are indicative of the goodness of the system.

Classification of Control Systems

- Consider the unity-feedback control system with the following open-loop transfer function

$$G(s) = \frac{K(T_a s + 1)(T_b s + 1)\cdots(T_m s + 1)}{s^N(T_1 s + 1)(T_2 s + 1)\cdots(T_p s + 1)}$$

- It involves the term s^N in the denominator, representing N poles at the origin.

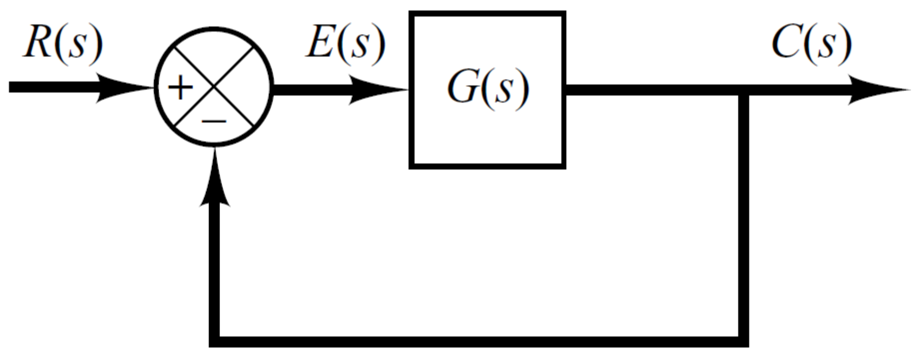
- A system is called type 0, type 1, type 2, ... , if N=0, N=1, N=2, ... , respectively.

Classification of Control Systems

- As the type number is increased, accuracy is improved.
- However, increasing the type number aggravates the stability problem.
- A compromise between steady-state accuracy and relative stability is always necessary.

Steady State Error of Unity Feedback Systems

- Consider the system shown in following figure.



- The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)} \qquad G(s) = \frac{K(T_a s + 1)(T_b s + 1)\cdots(T_m s + 1)}{s^N(T_1 s + 1)(T_2 s + 1)\cdots(T_p s + 1)}$$

Steady State Error of Unity Feedback Systems

- Steady state error is defined as the error between the input signal and the output signal when $t \rightarrow \infty$.

- The transfer function between the error signal E(s) and the input signal R(s) is $\frac{E(s)}{R(s)} = \frac{1}{1+G(s)}$
- The final-value theorem provides a convenient way to find the steady-state performance of a stable system.
- Since E(s) is $$E(s) = \frac{1}{1 + G(s)} R(s)$$
- The steady state error is

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)}$$

Static Error Constants

- The static error constants are figures of merit of control systems. The higher the constants, the smaller the steady-state error.
- In a given system, the output may be the position, velocity, pressure, temperature, or the like.
- Therefore, in what follows, we shall call the output "position," the rate of change of the output "velocity," and so on.
- This means that in a temperature control system "position" represents the output temperature, "velocity" represents the rate of change of the output temperature, and so on.

Static Position Error Constant (K_p)

- The steady-state error of the system for a unit-step input is

$$e_{ss} = \lim_{s \to 0} \frac{s}{1 + G(s)} \frac{1}{s}$$

$$= \frac{1}{1 + G(0)}$$

- The static position error constant K_p is defined by

$$K_p = \lim_{s \to 0} G(s) = G(0)$$

- Thus, the steady-state error in terms of the static position error constant K_p is given by

$$e_{ss} = \frac{1}{1 + K_p}$$

Static Position Error Constant (K_p)

- For a Type 0 system

$$K_p = \lim_{s \to 0} \frac{K(T_a s + 1)(T_b s + 1)\cdots}{(T_1 s + 1)(T_2 s + 1)\cdots} = K$$

- For Type 1 or higher order systems

$$K_p = \lim_{s \to 0} \frac{K(T_a s + 1)(T_b s + 1)\cdots}{s^N(T_1 s + 1)(T_2 s + 1)\cdots} = \infty, \qquad \text{for } N \geq 1$$

- For a unit step input the steady state error e_{ss} is

$$e_{ss} = \frac{1}{1 + K}, \qquad \text{for type 0 systems}$$

$$e_{ss} = 0, \qquad \text{for type 1 or higher systems}$$

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Static Velocity Error Constant (K_v)

- The steady-state error of the system for a unit-ramp input is

$$e_{ss} = \lim_{s \to 0} \frac{s}{1 + G(s)} \frac{1}{s^2}$$

$$= \lim_{s \to 0} \frac{1}{sG(s)}$$

- The static velocity error constant K_v is defined by

$$K_v = \lim_{s \to 0} sG(s)$$

- Thus, the steady-state error in terms of the static velocity error constant K_v is given by

$$e_{ss} = \frac{1}{K_v}$$

Static Velocity Error Constant (K_v)

- For a Type 0 system

$$K_v = \lim_{s \to 0} \frac{sK(T_a s + 1)(T_b s + 1)\cdots}{(T_1 s + 1)(T_2 s + 1)\cdots} = 0$$

- For Type 1 systems

$$K_v = \lim_{s \to 0} \frac{sK(T_a s + 1)(T_b s + 1)\cdots}{s(T_1 s + 1)(T_2 s + 1)\cdots} = K$$

- For type 2 or higher order systems

$$K_v = \lim_{s \to 0} \frac{sK(T_a s + 1)(T_b s + 1)\cdots}{s^N(T_1 s + 1)(T_2 s + 1)\cdots} = \infty, \qquad \text{for } N \geq 2$$

Static Velocity Error Constant (K_v)

- For a ramp input the steady state error e_{ss} is

$$e_{ss} = \frac{1}{K_v} = \infty, \qquad \text{for type 0 systems}$$

$$e_{ss} = \frac{1}{K_v} = \frac{1}{K}, \qquad \text{for type 1 systems}$$

$$e_{ss} = \frac{1}{K_v} = 0, \qquad \text{for type 2 or higher systems}$$

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Static Acceleration Error Constant (K_a)

- The steady-state error of the system for parabolic input is

$$e_{ss} = \lim_{s \to 0} \frac{s}{1 + G(s)} \frac{1}{s^3}$$

$$= \frac{1}{\lim_{s \to 0} s^2 G(s)}$$

- The static acceleration error constant K_a is defined by

$$K_a = \lim_{s \to 0} s^2 G(s)$$

- Thus, the steady-state error in terms of the static acceleration error constant K_a is given by

$$e_{ss} = \frac{1}{K_a}$$

Static Acceleration Error Constant (K_a)

- For a Type 0 system

$$K_a = \lim_{s \to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{(T_1 s + 1)(T_2 s + 1)\cdots} = 0$$

- For Type 1 systems

$$K_a = \lim_{s \to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{s(T_1 s + 1)(T_2 s + 1)\cdots} = 0$$

- For type 2 systems

$$K_a = \lim_{s \to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{s^2(T_1 s + 1)(T_2 s + 1)\cdots} = K$$

- For type 3 or higher order systems

$$K_a = \lim_{s \to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{s^N(T_1 s + 1)(T_2 s + 1)\cdots} = \infty, \qquad \text{for } N \geq 3$$

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Static Acceleration Error Constant (K_a)

- For a parabolic input the steady state error e_{ss} is

$$e_{ss} = \infty, \quad \text{for type 0 and type 1 systems}$$

$$e_{ss} = \frac{1}{K}, \quad \text{for type 2 systems}$$

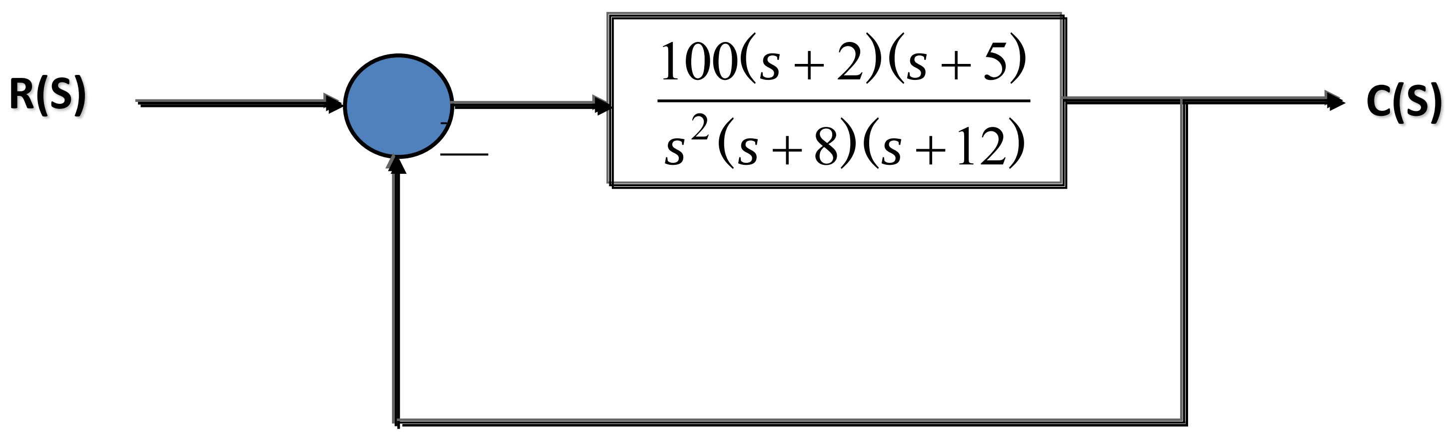
$$e_{ss} = 0, \quad \text{for type 3 or higher systems}$$

Summary

	Step Input $r(t) = 1$	Ramp Input $r(t) = t$	Acceleration Input $r(t) = \frac{1}{2}t^2$
Type 0 system	$\frac{1}{1 + K}$	∞	∞
Type 1 system	0	$\frac{1}{K}$	∞
Type 2 system	0	0	$\frac{1}{K}$

Example 2

- For the system shown in figure below evaluate the static error constants and find the expected steady state errors for the standard step, ramp and parabolic inputs.



Example 2

$$G(s) = \frac{100(s+2)(s+5)}{s^2(s+8)(s+12)}$$

$$K_p = \lim_{s\to 0} G(s)$$

$$K_p = \lim_{s\to 0}\left(\frac{100(s+2)(s+5)}{s^2(s+8)(s+12)}\right)$$

$$K_p = \infty$$

$$K_v = \lim_{s\to 0} sG(s)$$

$$K_v = \lim_{s\to 0}\left(\frac{100s(s+2)(s+5)}{s^2(s+8)(s+12)}\right)$$

$$K_v = \infty$$

$$K_a = \lim_{s\to 0} s^2 G(s)$$

$$K_a = \lim_{s\to 0}\left(\frac{100s^2(s+2)(s+5)}{s^2(s+8)(s+12)}\right)$$

$$K_a = \left(\frac{100(0+2)(0+5)}{(0+8)(0+12)}\right) = 10.4$$

Example 2

$K_p = \infty$ $K_v = \infty$ $K_a = 10.4$

$$e_{ss} = \frac{1}{1 + K_p} = 0$$

$$e_{ss} = \frac{1}{K_v} = 0$$

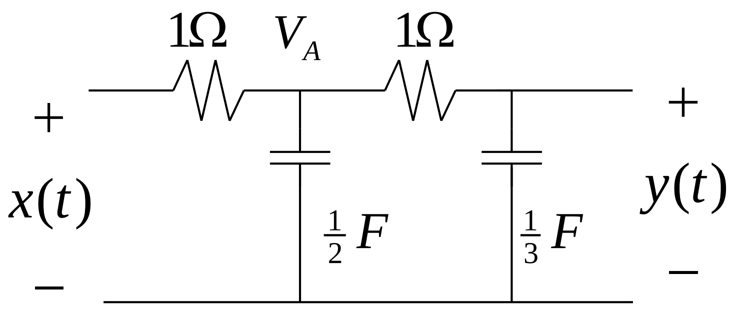
$$e_{ss} = \frac{1}{K_a} = 0.09$$

Stability

System representations

- Continuous-time LTI system
  - Ordinary differential equation
  - Transfer function (Laplace transform)
  - Dynamic equation (Simultaneous first-order ODE)
- Discrete-time LTI system
  - Ordinary difference equation
  - Transfer function (Z-transform)
  - Dynamic equation (Simultaneous first-order ordinary difference equation)

Continuous-time LTI system



$$\frac{V_A - x(t)}{1} + \frac{1}{2}\frac{dV_A}{dt} + \frac{V_A - y(t)}{1} = 0$$

$$\Rightarrow 2V_A + \frac{1}{2}\frac{dV_A}{dt} = x(t) + y(t)$$

$$\frac{y(t) - V_A}{1} + \frac{1}{3}\frac{dy(t)}{dt} = 0$$

$$\Rightarrow y(t) + \frac{1}{3}\frac{dy(t)}{dt} = V_A$$

$$2[y(t) + \tfrac{1}{3}y'(t)] + \frac{1}{2}[y'(t) + \tfrac{1}{3}y''(t)] = x(t) + y(t)$$

$$\Rightarrow \tfrac{1}{6}y''(t) + \tfrac{7}{6}y'(t) + y(t) = x(t)$$

$$\Rightarrow y''(t) + 7y'(t) + 6y(t) = 6x(t)$$

Ordinary differential equation

Laplace transform

$$s^2Y(s) + 7sY(s) + 6Y(s) = X(s)$$

$$H(s) \equiv \frac{Y(s)}{X(s)}\Big|_{I.C.=0} = \frac{6}{s^2 + 7s + 6}$$

Transfer function

$$y''(t) + 7y'(t) + 6y(t) = 6x(t)$$

$$let \quad \begin{aligned} x_1(t) &= y(t) \\ x_2(t) &= y'(t) \end{aligned}$$

$$\Rightarrow x_2'(t) + 7x_2(t) + 6x_1(t) = 6x(t)$$

$$\Rightarrow x_1'(t) = x_2(t)$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 6 \end{bmatrix} x(t)$$

State equation (Simultaneous first-order ODE)

Dynamic equation

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

output equation

Stability

- Internal behavior
  - The effect of all characteristic roots.
- External behavior
  - The effect by cancellation of some transfer function poles.

The Concept of Stability

A stable system is a dynamic system with a bounded response to a bounded input.

- Absolute stability is a stable/not stable characterization for a closed-loop feedback system.
- Given that a system is stable we can further characterize the degree of stability, or the relative stability.

<u>Definition</u> :

A system is internal (asymptotic) stable, if the zero-input response decays to zero, as time approaches infinity, for all possible initial conditions.

Asymptotic stable => All the characteristic polynomial roots are located in the LHP (left-half-plane)

Definition :

A system is external (bounded-input, bounded-output) stable, if the zero-state response is bounded, as time approaches infinity, for all bounded inputs.

bounded-input, bounded-output stable =>All the poles of transfer function are located in the LHP (left-half-plane)

Asymptotic stable => BIBO stable
BIBO stable≠> Asymptotic stable

System response

(i) First order system response

(ii) Second order system response

(iii) High order system response

The Routh-Hurwitz Stability Criterion

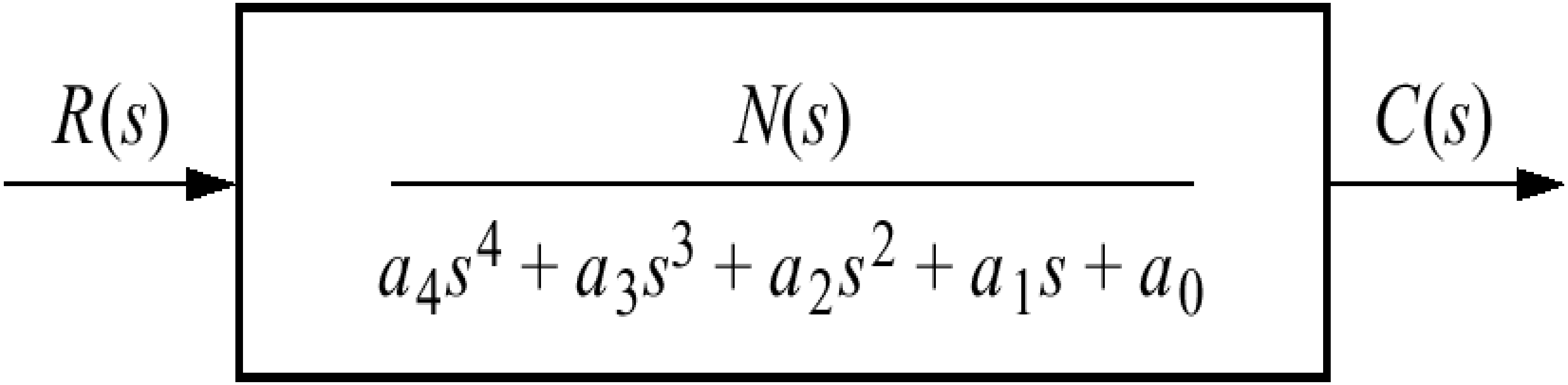
- It was discovered that all coefficients of the characteristic polynomial must have the same sign and non-zero if all the roots are in the left-hand plane.
- These requirements are necessary but not sufficient. If the above requirements are not met, it is known that the system is unstable. But, if the requirements are met, we still must investigate the system further to determine the stability of the system.
- The Routh-Hurwitz criterion is a necessary and sufficient criterion for the stability of linear systems.

The Routh-Hurwitz Stability Criterion Steps

The method requires two steps:

(1) Generate the data table (Routh table).

(2) Interpret the table to determine the number of poles in LHP and RHP.



s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2			
s^1			
s^0			

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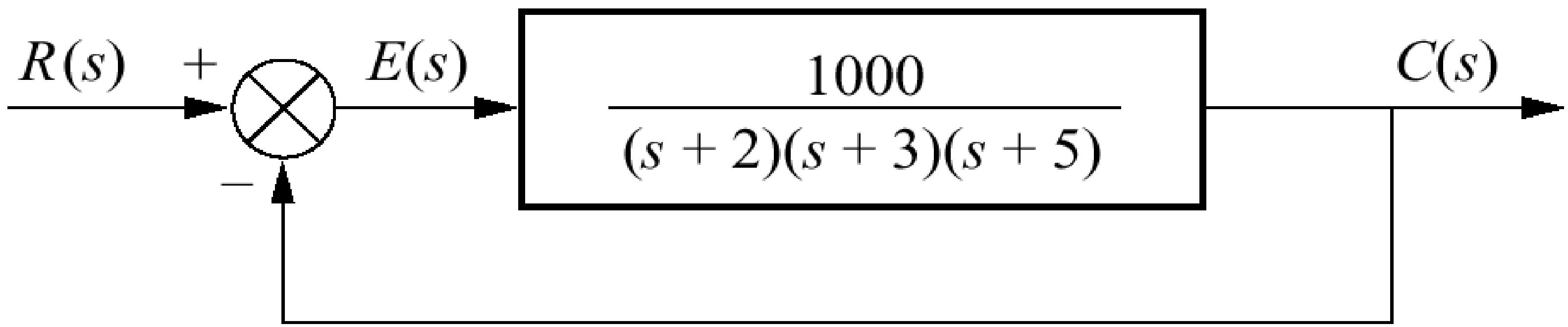
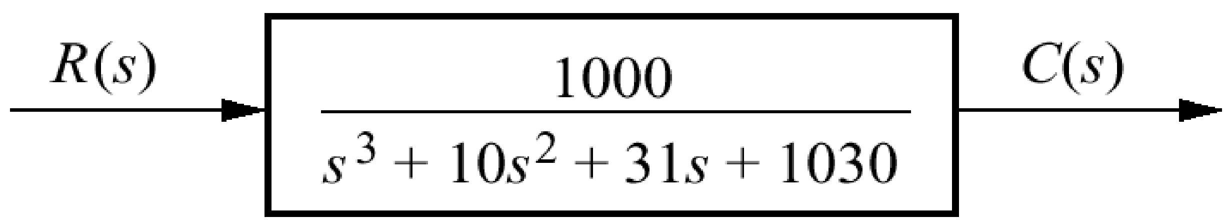
Completed Routh table

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\frac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\frac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\frac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\frac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$\frac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

Feedback system and its equivalent closed-loop system



(*a*)



(*b*)

Completed Routh table

s^3	1	31	0
s^2	~~10~~ 1	~~1030~~ 103	0
s^1	$\dfrac{-\begin{vmatrix} 1 & 31 \\ 1 & 103 \end{vmatrix}}{1} = -72$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0$
s^0	$\dfrac{-\begin{vmatrix} 1 & 103 \\ -72 & 0 \end{vmatrix}}{-72} = 103$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$

Interpretation of Routh table

The number of roots of the polynomial that are in the right half-plane is equal to the number of sign changes in the first column.

R-H: Special case. Zero in the first column

Replace the zero with ε , the value of ε is then allowed to approach zero from –ive or +ive side.

Problem Determine the stability of the closed-loop transfer function

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

s^5	1	3	5
s^4	2	6	3
s^3	~~0~~ ϵ	$\frac{7}{2}$	0
s^2	$\frac{6\epsilon - 7}{\epsilon}$	3	0
s^1	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	0	0
s^0	3	0	0

Label	First Column	$\epsilon = +$	$\epsilon = -$
s^5	1	+	+
s^4	2	+	+
s^3	~~0~~ ϵ	+	−
s^2	$\frac{6\epsilon - 7}{\epsilon}$	−	+
s^1	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	+	+
s^0	3	+	+

R-H: Special case. Entire row is zero

Problem Determine the number of right-half-plane poles in the closed transfer function $T(s) = \dfrac{10}{s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56}$

Solution: Form an auxiliary polynomial, P(s) using the entries of row above row of zeros as coefficient, then differentiate with respect to s finally use coefficients to replace the rows of zeros and continue the RH procedure.

$$P(s) = s^4 + 6s^2 + 8 \qquad \frac{dP(s)}{ds} = 4s^3 + 12s + 0$$

s^5	1	6	8
s^4	~~7~~ 1	~~42~~ 6	~~56~~ 8
s^3	~~0~~ ~~4~~ 1	~~0~~ ~~12~~ 3	~~0~~ ~~0~~ 0
s^2	3	8	0
s^1	$\frac{1}{3}$	0	0
s^0	8	0	0

Problem Determine the number of poles in the right-half-plane, left-half-plan and on the $j\omega$ axis for the closed transfer function

$$T(s) = \frac{20}{s^8 + s^7 + 12s^6 + 22s^5 + 39s^4 + 59s^3 + 48s^2 + 38s + 20}$$

s^8	1	12	39	48	20
s^7	1	22	59	38	0
s^6	$-\cancel{10}\ -1$	$-\cancel{20}\ -2$	$\cancel{10}\ 1$	$\cancel{20}\ 2$	0
s^5	$\cancel{20}\ 1$	$\cancel{60}\ 3$	$\cancel{40}\ 2$	0	0
s^4	1	3	2	0	0
s^3	$\cancel{0}\ \cancel{4}\ 2$	$\cancel{0}\ \cancel{6}\ 3$	$\cancel{0}\ \cancel{0}\ 0$	0	0
s^2	$\cancel{\frac{3}{2}}\ 3$	$\cancel{2}\ 4$	0	0	0
s^1	$\frac{1}{3}$	0	0	0	0
s^0	4	0	0	0	0

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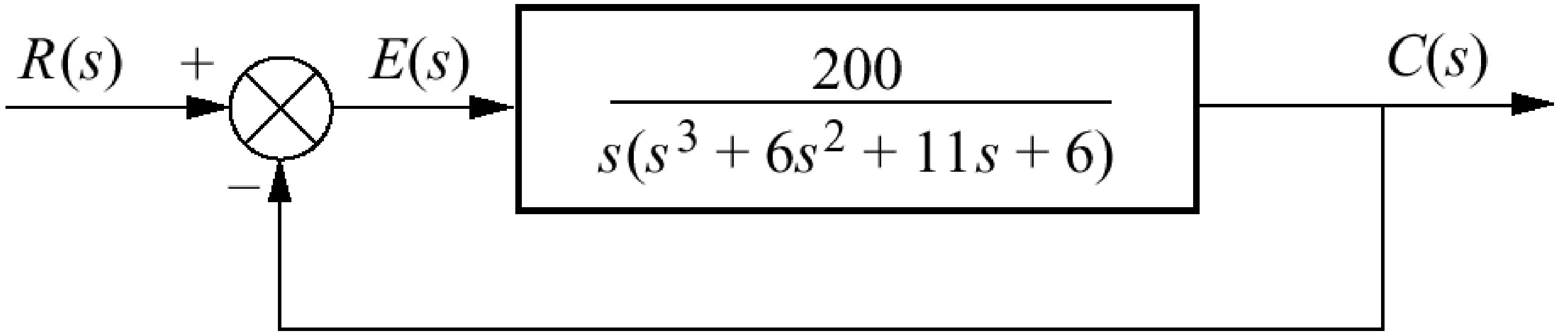
Problem Determine the number of poles in the right-half-plane, left-half-plan and on the $j\omega$ axis for system



Solution: The closed loop transfer function is

$$T(s) = \frac{200}{s^4 + 6s^3 + 11s^2 + 6s + 200}$$

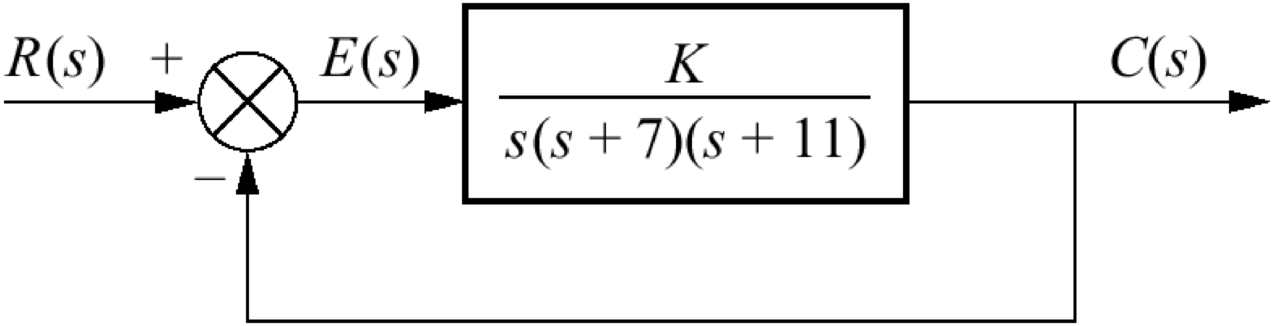
Routh table

s^4	1	11	200
s^3	~~6~~ 1	~~6~~ 1	
s^2	~~10~~ 1	~~200~~ 20	
s^1	-19		
s^0	20		

2 poles in RHP, 2 poles in LHP no poles on $j\omega$ axis
The system is unstable

Feedback control system

Problem Find the range of gain K for the system that will cuase the system to be stable, unstabel, and marginally stable. Assume K>0.



Solution: The closed loop transfer function is

$$T(s) = \frac{K}{s^3 + 18s^2 + 77s + K}$$

Routh table for Example 6.9

s^3	1	77
s^2	18	K
s^1	$\dfrac{1386-K}{18}$	
s^0	K	

s^3	1	77
s^2	18	1386
s^1	~~0~~ 36	
s^0	1386	

For K < 1386 the system is stable. For K > 1386 the system is unstable. For K = 1386 we will have entire row of zeros (s row). We form the even polynomial and differentiate and continue, no sign changes from the even polynomial so the 2 roots are on the $j\omega$ axis and the system is marginally stable

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Routh table for Example 6.10

Problem Factor the polynomial $s^4 + 3s^3 + 30s^2 + 30s + 200$

Solution: from the Routh table we see that the s^1 row is a row of zeros. So the even polynomial at the s^2 row is $P(s) = s^2 + 10$ **since this polynomial is a factor of the original, dividing yields** $P(s) = s^2 + 3s + 20$ **As the other factor so**

$$s^4 + 3s^3 + 30s^2 + 30s + 200 = (s^2 + 10)(s^2 + 3s + 20)$$

s^4	1	30	200
s^3	~~3~~ 1	~~30~~ 10	
s^2	~~20~~ 1	~~200~~ 10	
s^1	~~0~~ 2	~~0~~ 0	
s^0	10		

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Stability is State Space Example 6.11

Problem **Given the system**

$$\dot{X} = \begin{bmatrix} 0 & 3 & 1 \\ 2 & 8 & 1 \\ -10 & -5 & -2 \end{bmatrix} X + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} X$$

Find out how many poles in the LHP, RHP and on the $j\omega$ **axis**

Solution: **First form (sI-A)**

$$(sI - A) = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 3 & 1 \\ 2 & 8 & 1 \\ -10 & -5 & -2 \end{bmatrix} = \begin{bmatrix} s & -3 & -1 \\ -2 & s-8 & -1 \\ 10 & 5 & s+2 \end{bmatrix}$$

Now find the det (sI-A) = $s^3 - 6s^2 - 7s - 52$

Routh table for Example 6.11

s^3	1	-7
s^2	$-\cancel{6}\ \ -3$	$-\cancel{52}\ \ -26$
s^1	$-\cancel{\frac{47}{3}}\ \ -1$	$\cancel{0}\ \ 0$
s^0	-26	

One sign change, so 1 pole in the LHP and the system is unstable

With Our Best Wishes

Automatic Control (1)

Course Staff

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Thank You
For Your Attention

Mohamed Ahmed Ebrahim

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